CONVENTIONS AND CONTEXT: GRAPHING RELATED OBJECTS ONTO THE SAME SET OF AXES

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Several researchers have promoted reimagining functions and graphs more quantitatively. One part of this research has examined graphing "conventions" that can at times conflict with quantitative reasoning about graphs. In this theoretical paper, we build on this work by considering a widespread convention in mathematics teaching: putting related, derived graphical objects (e.g., the graphs of a function and its inverse or the graphs of a function and its derivative) on the same set of axes. We show problems that arise from this convention in different mathematical content areas when considering contextualized functions and graphs. We discuss teaching implications about introducing such related graphical objects through context on separate axes, and eventually building the convention of placing them on the same axis in a way that this convention and its purposes become more transparent to students.

Keywords: Mathematical Representations, High School Education, Undergraduate Education

Graphs are a foundational mathematical construct used ubiquitously across science, technology, engineering, and mathematics (STEM). Modeling quantitative relationships through graphs has been promoted as essential in STEM education (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Research Council, 2012). Yet, there are two important issues that lie at the intersection of graphs and modeling contexts. First, there is a tension between (a) the power of *contextualizing* mathematics for conceptual understanding and quantitative reasoning and (b) the power of *abstraction* in mathematics to see general structures and underlying ideas (Freudenthal, 1968; Mitchelmore & White, 2007; van den Heuvel-Panhuizen, 2003). Second, there is a tension between (a) using conventions in displaying graphs for communicative or illustrative purposes, and (b) conflicts those conventions may have with perceiving and understanding deeper quantitative relationships (Moore & Silverman, 2015; Moore et al., 2014; Moore et al., 2019).

Previous research work, discussed in the next section, has made strides in bringing quantitative and covariational reasoning to bear on students' graphical thinking. In this theoretical paper, we contribute to this area by examining the convention of graphing related, derived objects (e.g., the graphs of a function and its inverse, or the graphs of a function and its derivative, or input and output vectors) onto the same set of axes in connection with contextualization versus abstraction. To do so, we use example cases from across different mathematical areas to describe how this convention, which is appropriate for abstract situations, conflicts with contextualization and quantitative reasoning. We then offer implications that our theoretical exploration has for teaching mathematical topics involving this graphing convention.

Literature Review

Graphs play a central role in representing quantities and quantitative situations mathematically, and they are used extensively across STEM fields to model a wide variety of phenomena (e.g., Angra & Gardner, 2017; Beichner, 1996; Planinic et al., 2012; Rodriguez, Bain, Towns, et al., 2019; Rodriguez, Bain, & Towns, 2019). Unfortunately, research from both

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mathematics and science education shows clearly that students have difficulties using graphs with quantitative situations (e.g., Bajracharya & Thompson, 2014; Beichner, 1994; McDermott et al., 1987; Testa et al., 2002; Woolnough, 2000). One prominent tendency is for students to see graphs as "pictures" or "shapes" rather than as a depiction of a relationship between quantities (Beichner, 1994; Leinhardt et al., 1990; McDermott et al., 1987; Moore & Thompson, 2015). Other challenges include confusing "slope" with "height" (Hale, 2000; McDermott et al., 1987; Planinic et al., 2012), making incorrect assertions based on the visual look of a graph (Aspinwall et al., 1997), understanding changing rates (Carlson et al., 2002), and making local versus global interpretations (Leinhardt et al., 1990; Monk, 1994). One key issue is that many conventions, such as x residing on the horizontal axis or a *vertical* line test, end up being considered as essential requirements for graphs by students (Moore & Silverman, 2015; Moore et al., 2019).

In order to promote better understanding, several researchers have focused on conceptually-rich and quantitatively-founded ways of approaching graphs in mathematics education. Some early work in this area focused on using videos and computers to track information such as distance and velocity to produce graphs (Beichner, 1996; Zollman & Fuller, 1994), or using motion detectors to embody graphical activity (Beckmann & Rozanski, 1999; Berry & Nyman, 2003). Since then, a significant portion of research has focused on re-imagining functions and graphs at the fundamental level through covariation (e.g., Carlson et al., 2002; Castillo-Garsow et al., 2013; Ellis et al., 2016; Moore & Thompson, 2016; Paoletti & Moore, 2018; Thompson & Carlson, 2017). Several studies have reported on improved student understanding and reasoning about functions, graphs, and coordinate systems when students develop these covariation-based ways of thinking (Ellis, 2011; Ellis et al., 2016; Moore, 2014; Moore et al., 2013).

In particular, Moore and Thompson (2015) contrast *static thinking*, in which students think of the graph as a static object (like a "wire"), with *emergent thinking*, in which students imagine the covariational relationship between x and y as tracing out the graph. In emergent thinking, a graph involving quantities communicates an evolving "story" between the quantities (Rodriguez, Bain, Towns, et al., 2019). However, a barrier to thinking this way is students' adoption of certain conventions as being necessary for graphs to be mathematically correct. For example, Moore et al. (2019) explain that students state that a sine curve snaking up the y-axis does not suggest a function, because of the vertical line test, despite the fact that the graph perfectly well represents the functional relationship $x = \sin(y)$. Moore and colleague's work suggests that confronting these conventions directly can help students develop a stronger sense of how graphs can portray quantitative relationships (Moore et al., 2014; Moore et al., 2019). Our paper builds on the existing literature by considering an important convention in mathematical graphing activity, described in the next section, and its conflict with representing quantitative relationships.

Theoretical Perspective: Conventions in Graphing

This study uses the lens of "conventions" in terms of graphical activity. Hewitt (1999, 2001a, 2001b) described what he called "arbitrary" aspects of mathematics, consisting of social conventions that do not necessarily have to be done that way, but on which some consensus has been reached. Some examples Hewitt provided were the names of shapes, the usage of the symbols x and y as coordinates, or terminology for operations (1999, p. 4). Other examples in the context of graphing could include using perpendicular axes (unlike Descartes' early conventions, Katz, 2009), having "up" be the positive direction, or using uniform scaling. Thompson (1992) proposed the importance of students becoming aware of these conventions they were using. He used the phrase "convention qua convention" to mean when one understands "that approaches

other than the one adopted could be taken with equal validity" (p. 125). He explained that problems arise when conventions are not properly understood as conventions (see also Thompson, 1995). Moore et al. (2019) then built on these ideas by providing a definition of "convention," which we use in this paper: a convention is a combination of a concept, a community, and a representational practice.

The precise convention we highlight in this paper is the common practice of placing a graphical object and a related, derived graphical object on the same set of axes. Here, by "object" we mean the literal "thing" that is placed on the axes, such as a function's graph, a curve, or a vector arrow. Applying Moore et al.'s (2019) definition, we see the "community" as the mathematical community, the "concept" as graphical objects related in some key way, and the "practice" as placing these related objects on the same set of axes. As examples, it is common to graph a function and its inverse function on the same set of axes (e.g., Blitzer, 2018; Sullivan & Sullivan, 2020), or to place the graphs of a function and its derivative on the same set of axes (e.g., Hass et al., 2020; Rogawski et al., 2019; Stewart et al., 2021).

Conventions in mathematics also have connection to abstraction in mathematics (Dreyfus, 2020; Ferrari, 2003). A core practice of mathematics is abstracting similar structures from different contexts (Brousseau, 1997, 2002). This practice results in decontextualized objects, for which certain conventions might be adopted to track them in the absence of concrete quantities, such as using x primarily as the "input/independent" variable, or placing the output primarily on the vertical axis. Such conventions in communicating about abstract structures may be perfectly valid in the abstract space, but may conflict with reasoning within a quantitative context.

The goal of this theoretical paper is to examine cases of this convention across mathematics content areas to show how it may problematically intersect with contextual graphs that represent quantities. We explain how not understanding this convention *as* a convention can lead to issues in making sense of contextualized or quantitative interpretations of a graphical system. We then describe teaching implications based on our theoretical investigation and attempt to situate this convention more appropriately within learning about representing related graphical objects.

Conflicts between the Convention and Contextualization in Case Content Areas

In this section, we unpack the convention described in the previous section in three distinct mathematical content area cases. The primary purpose of this section is to show, theoretically, the conflicts that can arise between this particular convention and contextualization. The next section then discusses the pedagogical implications that may help address this conflict.

Case 1: Graphing Inverse Functions

One key concept in mathematics is inverse functions. Students are exposed to inverse functions each time they learn about a new function (e.g., exponential, trigonometric). A common convention of inverse functions is to graph the function and its inverse on the same set of axes, where the inverse is a reflection of the original graph over the $y = x$ line. Figure 1 shows common types of images students see in textbooks and in classrooms. In abstraction, this convention can make sense to examine how the features of the two graphs relate to each other.

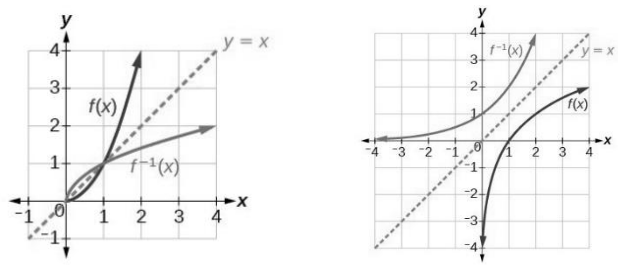

Figure 1: Common types of images of functions and inverses on the same set of axes

The problem with this convention is in moving to a contextualized situation. For example, if one examines a function converting Euros to U.S. Dollars, D(E) = 1.3E (Teuscher et al., 2018), one might create a graph with Euros on the horizontal axis and U.S. Dollars on the vertical axis. The point (50,65) on the graph suggests that an input of 50€ gives an output of $65. However, if the inverse function is graphed as a reflection across the D = E line on the same axes, then the new graph conflicts with the quantitative meanings of the two axes. The point (50,65) reflects to the point (65,50), now suggesting, according to the axes, that 65€ corresponds to $50, which is incorrect (Figure 2). To accurately interpret the placement of the two graphs on the same axes, a student must make *both* axes simultaneously represent *both* the D and E quantities, tracking which version of the axis label matches which graph is being examined. For novice learners of inverse functions, this may be quite sophisticated reasoning, obscuring the actual quantitative relationship that exists for a function and its inverse (see also Duval, 2016). Thus, the convention of placing two related graphical objects (graphs of a function and its inverse) on the same axes may conflict with contextualizing the functions and graphs through quantities.

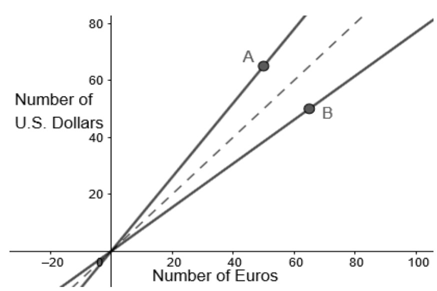

Figure 2: Graphs of function and inverse in the context of money

Case 2: Graphing Derivatives of Functions

This convention also appears in the practice of placing the graph of a function and the graph of its derivative on the same axes. Calculus textbooks and instructors often do this, as exemplified in Figure 3 taken from Stewart et al. (2021, p. 164) and Hughes-Hallett et al. (2012, p. 106). If the variables x and y represent nothing more than pure numbers, this practice is appropriate, since one can put the numeric outputs of f and f' on the same *unit-less* axes.

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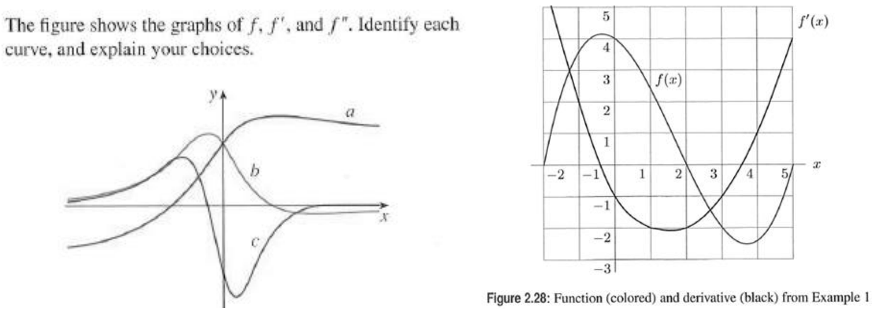

Figure 3: Function and derivative graphs on the same axes

As with inverse functions, the problem arises when trying to contextualize these functions. For example, suppose $f(x)$ represents the temperature in degrees Celsius as a function of time, with x in units of minutes. Then the output of the derivative, f', is the rate of change of temperature in degrees per minute, which is a fundamentally different quantity than the output of f as temperature. When placing f and f' on the same axes, some confusion arises: What quantity is represented by the vertical axis? Is it the temperature? The rate of change? Both? In fact, it would require *both* quantities being on the vertical axis simultaneously. For example, Figure 4, taken from Hass et al. (2020, p. 150), shows a position function (s) and its derivative function (velocity, v). Note the label "*s, v*" on the vertical axis, meaning that single axis is simultaneously representing two quantities. When looking at $s = 5\cos(t)$ or $v = -5\sin(t)$, one must constantly shift the vertical axis between position and velocity quantities and their associated units. While experts can likely make such subtle shifts without much problem, and it may even be a goal for students to eventually do such thinking, this convention again conflicts with contextualizing functions and their derivatives and examining them quantitatively, especially for novices.

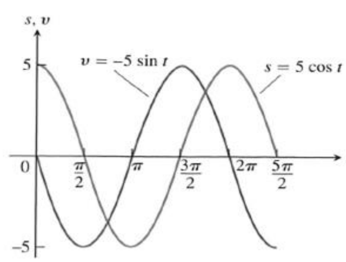

Figure 4: The vertical axis representing two quantities simultaneously

Case 3: Graphing Matrix Operations

One common way to think about matrix-vector multiplication in linear algebra is as a matrix performing a geometric transformation on a vector. In this perspective it is common to show the original vector and the transformed vector on the same set of axes (e.g., Poole, 2015), as in Figure 5. When working in the abstract world of vectors as geometric arrows or lists of unit-less numbers, this convention can help to illustrate the transformation that is represented by a particular matrix.

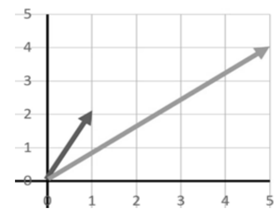

Figure 5: A vector and a transformed vector on the same set of axes

However, yet again, contextualizing vectors and matrices can possibly come into conflict with this convention. As an example, the second author collaborated with two secondary teachers to develop a set of tasks to help students better understand the structure of matrix multiplication and how matrix multiplication can be used to model real-world phenomena. The tasks used the context of two basketball players, Joaquin and Raul, who played the same position but averaged different numbers of points and rebounds each quarter, with Joaquin averaging 6 points and 3 rebounds per quarter and Raul averaging 2 points and 5 rebounds per quarter. The students were asked: The basketball coach wants his centers to combine for 20 points and 14 rebounds, because that might help them win the game. What number of quarters could the coach play Joaquin and Raul to average 20 points and 14 rebounds? This problem gives rise to the matrix equation: $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} 20 \\ 14 \end{bmatrix}$, with solution $\begin{bmatrix} J \\ R \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

From a mathematical standpoint, we can think about this as a 2-by-2 matrix that transforms the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to the vector $\begin{bmatrix} 20 \\ 14 \end{bmatrix}$. However, the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is in units of *quarters played* for each player, $\begin{bmatrix} \text{1 J quarter} \\ \text{3 R quarters} \end{bmatrix}$, while the output vector has units of *points* and *rebounds*: $\begin{bmatrix} \text{20 points} \\ \text{14 rebounds} \end{bmatrix}$. When plotting these two vectors on the same axes, we again see a complication with what quantities the axes represent. The horizontal axis must simultaneously represent “Joaquin quarters” *and* “points” while the vertical axis must simultaneously represent “Raul quarters” *and* “rebounds” (Figure 6). Like with function inverses and derivatives, this is likely difficult for novices learning about matrix-vector multiplication, and we see that the convention of placing related graphical objects on the same axes once again can be at odds with contextualization.

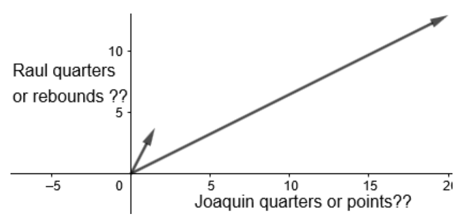

Figure 6: Conflict in the Quantities on Each Axis

When the Convention Does *Not* Conflict with Contextualization

To be clear, the convention of placing related, derived graphical objects on the same axes does not *always* conflict with contextualization. For example, consider certain graphical transformations, such as a vertical shift given by $f(x) + 4$ or a vertical stretch given by $2f(x)$ (Figure 7). If the function and graph represent a quantitative context, such as x representing the price of a good and $f(x)$ representing the amount sold of that good, then the transformations typically retain those same quantitative meanings. The function $f(x) + 4$ can represent that an

additional fixed 4 units are sold at all price levels, and $2f(x)$ can represent that twice as many units are sold at each price level. In both of these cases, the vertical axis retains the same quantity "amount sold" and there is no conflict with the convention and the context.

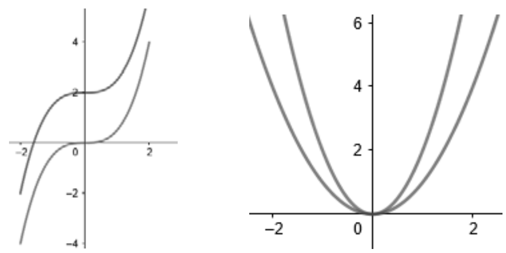

Figure 7: Transformations graphed on the same axes

Teaching Implications

We have explained how the practice of placing related, derived graphical objects on the same axes can often conflict with using quantitative contexts for functions and graphs. The answer should not be to avoid contextualization to sidestep the conflict and teach solely with abstract pure-numeric functions (Ferrari, 2003). But, it also cannot be to avoid the abstract convention, because it is a common practice. To address the dilemma, instruction should use context, but then help students see the convention *as* a convention (Moore et al., 2019; Thompson, 1992). To this end, we offer the following suggestions, based on Brousseau's (1997; 2002) idea of contextualizing mathematics in order to then re-decontextualize it. Thus, we take the stance that teaching using contextualized quantitative situations is crucial, but that students must be guided toward a comprehension and usage of the abstract convention we have described.

The first step must involve the teacher identifying whether a mathematical topic wherein two related graphical objects are commonly placed on the same axes actually conflicts with contextualization, or not, as we did in the previous section. Thus, the previous section contributes as a model for that type of conceptual analysis. If a potential conflict is identified, we recommend that the graphical representations of the two related objects initially be placed on *separate* axes. To use inverse functions as an example, if the class is learning that the function D(E) = 1.3E has an inverse function E(D) = D/1.3, then the two graphs should be placed initially on two separate axes (Figure 8). Doing so initially avoids the conflict between quantitative reasoning and using a single set of axes and permits the students to track how each graphical object (function or inverse, in this case) relates to the context's quantities. With each graph separate, students can develop quantitative reasoning about the overall context, without the difficulties described in the previous section, such as identifying the rate that one currency accumulates as the other currency accumulates (see also Moore, 2016; Moore & Thompson, 2015; Thompson & Carlson, 2017). Thus, in the first step, the focus is on quantitatively understanding each mathematical object *in its own right*, at the global level.

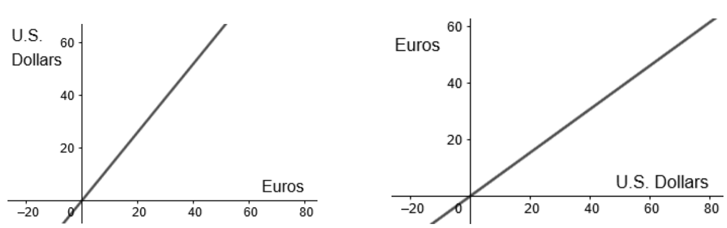

Figure 8: First, Begin with Related Objects on Separate Axes

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The second step acts as an intermediate between the initial contextualization and the abstracted, decontextualized convention. In this step, students are asked to compare the two graphical objects to identify specific connections, differences, or relationships. A key part of this step is to explicitly ask the students to place the two sets of axes near each other in a way that facilitates comparisons between the two graphical objects. For example, suppose students are investigating the graphs of a function and its derivative, where $f(x)$ represents the temperature as a function of time. As per step one, the students have represented the graphs of f and f' on separate axes and have examined the behaviors of the temperature and the rate of change separately. The teacher could ask, "Compare the graph of f with the graph of f'. What connections do you see between them? I encourage you to put the derivative graph directly underneath the function graph to help compare them." Such placement allows some connections to become clearer, such as what behavior for f is associated with f' being positive, negative or zero (Figure 9). The key feature of this step is that, while still on separate axes, the students are now thinking about the benefits of placing the graphs more closely to each other. The physical placement gets the students one step closer to the convention, while still maintaining a focus on the quantitative relationships.

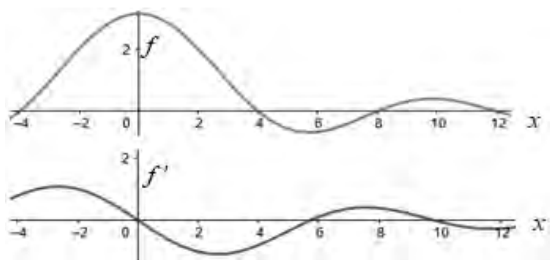

Figure 9: Second, Place Graphs in a Way to Examine Specific Connections

The third step then moves to the abstract convention and its purposes, while explicitly attending to quantitative difficulties associated with the convention. In this step, the teacher can place the two graphical objects on the same axes and ask the students what the advantages and disadvantages of doing so might be. To use matrix-vector multiplication as an example, suppose a teacher has used the quarters-played / points-rebounds context, has started with the vectors on two different axes (step one), and has had the students compare the two vectors by placing the two axes close to each other (step two). The teacher then places the two vectors on the same axes, and asks, "What might be the benefit of putting the two vectors on the same axes? But what might be confusing about doing that?" This step helps students identify the benefits of seeing directly how one vector can be thought of as a transformed version of the other. Yet, the discussion also helps students see the difficulties in thinking about one vector as representing one pair of quantities (quarters played) while the other vector represents a different pair of quantities (points and rebounds). By now seeing this convention *as* a convention (Thompson, 1992), the students can see why it is used in the first place, when they might choose to use it, and what cognitive work they need to do to make sure they can quantitatively interpret graphs superimposed on top of each other.

In conclusion, we believe these three steps address the problems described in the previous section by allowing the concepts to be taught with contextualization and quantitative reasoning, but then gradually transitioning to the abstract convention (Brousseau, 1997; 2002). Such a sequence of steps makes the convention of placing the graphical objects on the same axes more

reasonable, but also helps students be aware of the challenges associated with it. We believe these steps put students in a better position to develop the sophisticated abstract thinking that goes along with assigning multiple (conflicting) quantities on the same axis. Students can use this convention appropriately both within and outside mathematics. That is, not only can they better appreciate the convention for abstract graphical objects, but they can also better reason about contexts that *are* represented with multiple quantities on the same axis, such as the position and velocity graphs in Figure 4, or the "double *y*-axis" graphs shown here in Figure 10.

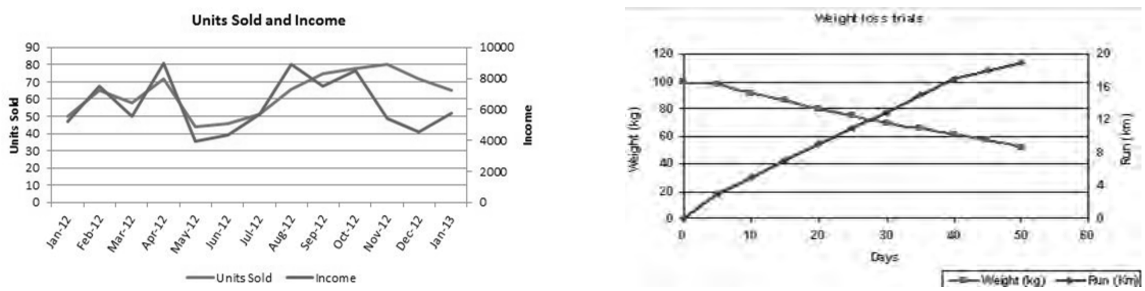

Figure 10: Double *y*-axis Graphs with Two "Vertical Axis" Quantities

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